Journal of Statistical Physics, Vol. 124, Nos. 2-4, July/August 2006 (© 2006)
DOI: 10.1007/s10955-006-9035-9

Decay Rates in Probability Metrics Towards Homogeneous Cooling States for the Inelastic Maxwell Model

M. Bisi,[1] J. A. Carrillo,[2] and G. Toscani[3]

Received 22 September 2005; accepted 31 January 2006
Published Online: March 28, 2006

We quantify the long-time behavior of solutions to the nonlinear Boltzmann equation for spatially uniform freely cooling inelastic Maxwell molecules by means of the contraction property of a suitable metric in the set of probability measures. Existence, uniqueness, and precise estimates of overpopulated high energy tails of the self-similar profile proved in ref. 9 are revisited and derived from this new Liapunov functional. For general initial conditions the solutions of the Boltzmann equation are then proved to converge with computable rate as $t \to \infty$ to the self-similar solution in this distance, which metrizes the weak convergence of measures. Moreover, we can relate this Fourier distance to the Euclidean Wasserstein distance or Tanaka functional proving also its exponential convergence towards the homogeneous cooling states. The findings are relevant in the understanding of the conjecture formulated by Ernst and Brito in refs. 15, 16, and complement and improve recent studies on the same problem of Bobylev and Cercignani(9) and Bobylev, Cercignani and one of the authors.(11)

KEY WORDS: inelastic collisions, Maxwell models, asymptotic behavior, Fourier metrics, Ernst-Brito conjecture

1. INTRODUCTION

This paper concerns the large time behavior of solutions of the homogeneous Boltzmann equation for the inelastic Maxwell molecules introduced in ref. 7

$$\frac{\partial f}{\partial t} = B\sqrt{\theta(t)}Q(f, f)\ . \tag{1.1}$$

[1] Dipartimento di Matematica, Università di Parma, Parco Area delle Scienze 53/A, I-43100 Parma, Italy; e-mail: marzia.bisi@unipr.it
[2] ICREA-Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain; e-mail: carrillo@mat.uab.es
[3] Dipartimento di Matematica, Università di Pavia, via Ferrata 1, I-27100 Pavia, Italy; e-mail: giuseppe.toscani@unipv.it

Here, $f(v,t)$ is the density for the velocity space distribution of the molecules at time t, while $Q(f,f)$ is the inelastic Boltzmann collision operator, which contains the effects of binary collisions of grains. In expression (1.1), the factors B and the temperature of f in front of Q,

$$\theta(t) = \frac{1}{3}\int_{\mathbb{R}^3} |v|^2 f(v,t)\,dv,$$

allow the Maxwell model to have the same loss of temperature law of the inelastic hard-spheres model.[7] The collision operator $Q(f,f)$ is more easily treated if expressed in weak form. This corresponds to write, for every suitable test function φ,

$$(\varphi, Q(f,f)) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)f(w)\left[\varphi(v') - \varphi(v)\right] dv\,dw\,dn. \qquad (1.2)$$

In (1.2) v' is the outgoing velocity assumed by a particle in the collision defined by the ingoing velocities v, w and the angular parameter $n \in S^2$:

$$v' = \frac{1}{2}(v+w) + \frac{1-e}{4}(v-w) + \frac{1+e}{4}|v-w|n,$$
$$w' = \frac{1}{2}(v+w) - \frac{1-e}{4}(v-w) - \frac{1+e}{4}|v-w|n.$$

The constant $0 < e < 1$ is the normal restitution coefficient.

Inelastic Maxwell models are of interest for granular fluids in spatially homogeneous states because of the mathematical simplifications resulting from their energy-independent collision rate. For this reason, after its introduction in ref. 7, Eq. (1.1) has been widely studied with or without energy supplies.

Easy computations show that $(\varphi(v), Q(f,f)) = 0$ whenever $\varphi(v) = 1, v$, while $(\varphi(v), Q(f,f)) < 0$ if $\varphi(v) = v^2$. This corresponds to conservation of mass and momentum, and, respectively, to loss of energy for the solution to Eq. (1.1). For this reason, if we fix the initial data to be a centered probability density function, the solution will remain centered at any subsequent time $t > 0$.

It is well-known[6] that both elastic and inelastic Maxwell models allow one to take advantage of the powerful Fourier transform methods. In the elastic case, the importance of working in Fourier spaces with suitable Fourier metrics has been first remarked in ref. 17.

Let $\mathcal{P}_s(\mathbb{R}^3)$ be the set of probability measures with bounded s-moment. For any pair of probability measures in $\mathcal{P}_s(\mathbb{R}^3)$, the Fourier-based metrics d_s, for any $s > 0$, are defined as

$$d_s(\hat{f}, \hat{g}) = \sup_{k\in\mathbb{R}^3} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^s}.$$

As usual, $\hat{f}$ is the Fourier transform of the density $f(v)$,

$$\hat{f}(k,t) = \int_{\mathbb{R}^3} f(v,t) \mathrm{e}^{-iv\cdot k} dv.$$

By simple Taylor expansion, one shows that the distance is well-defined and finite for any pair of probability measures with equal moments up to order $[s]$, where $[s]$ denotes the integer part of s. Moreover, in case $s \geq 1$ be an integer, it suffices equality of moments up to order $s-1$ for being d_s finite. In fact, d_s with $s \geq 2$ topology is equivalent to the weak-star topology for measures plus convergence of moments up to order $[s]$,[21] and can be related to the Wasserstein distance between probability measures.

Among others, one of the interesting features of granular flows, which can be observed in the framework of Maxwellian molecules, is the existence of self-similar solutions in the homogeneous cooling problem, and the non-Maxwellian behavior of these solutions, which display power-like decay for large velocities. Self-similar solutions of Eq. (1.1) are obtained through a suitable scaling of both time and velocity (cfr. Sec. 2) in such a way that energy of the solution is conserved. If

$$f(v,t) = \theta^{-\frac{3}{2}}(\tau) g\big(v\theta^{-\frac{1}{2}}(\tau), \tau\big),$$

where

$$\tau = \frac{B}{E} \int_0^t \theta^{\frac{1}{2}}(w)\, dw,$$

and $E = 8/(1-e^2)$, g satisfies the equation

$$\frac{\partial g}{\partial \tau} = -\nabla_v \cdot (vg(v)) + EQ(g,g). \tag{1.3}$$

Self-similar solutions (homogeneous cooling states) of the original Boltzmann equation correspond to stationary solutions g_∞ of Eq. (1.3). Existence of stationary solutions with power-like tails has been proven by several authors.[1–3, 15] A systematic approach to the existence of self-similar profiles for both elastic and inelastic interactions was subsequently proposed by Bobylev and Cercignani in ref. 9, who obtained also results of convergence towards the self-similar solution. Later on, these results have been improved in ref. 11, by showing that convergence towards the self-similar profile occurs for all solutions corresponding to initial data which have more than two moments bounded.

Concerning the problem of the Boltzmann equation with an energy source, Bobylev and Cercignani[8] found steady solutions to the inelastic Maxwell model with a heat bath, that behave like $\exp(-r|v|)$. The problem of convergence towards the steady solution has been subsequently dealt with in ref. 4. By means of the strict contraction property of d_2 in $\mathcal{P}_2(\mathbb{R}^3)$, existence, uniqueness, boundedness of moments and regularity of the steady state have been derived. Furthermore, explicit decay rates of general solutions towards the stationary state were obtained in all Sobolev spaces with a rate of decay as close as the one obtained for the contraction estimate.

The main aim of this paper is to improve in the understanding of the nonlinear stability and convergence towards equilibrium for the self-similar problem (1.3) based on the techniques introduced in ref. 4 for the inelastic problem and in ref. 17 for Maxwellian molecules in the classical elastic case. We first remark that the distance d_2 is proved to be a non-strict contraction for the flow. This is a direct consequence of the main Theorem in ref. 4. Thus, in order to get rates of convergence we will show a strict contraction property for $d_{2+\alpha}$, $0 < \alpha < \bar{G}(e)$ with $\bar{G}(e) > 2$.

Contraction properties of $d_{2+\alpha}$ allow us to revisit the existence and uniqueness of the similarity solution, as well as various properties of the solution itself. In particular, it will be possible to discuss in detail the conjecture on the self-similar solution formulated by Ernst and Brito in refs. 15, 16, namely tail behavior and rates of convergence towards equilibrium.

Concerning tail behavior: second part of the EB conjecture, we show that the facts of the flow being contractive with Fourier distances and the homogeneous cooling state to have a corresponding moment bounded are equivalent. More precisely the critical exponent on the contraction estimate $\bar{G}(e) > 2$ corresponds exactly to the critical exponent for the unboundedness of the moment of the homogeneous cooling state. Then, this gives a new, we believe, elegant interpretation of the "thick tails" results proven for the first time in ref. 9. Moreover, we will make use of this fact to show uniform in time propagation of moments for solutions of the Cauchy problem to (1.3) which has not been addressed before.

Concerning rates of decay: first part of the EB conjecture, although the rate of decay in the first result in Sec. 3 coincides with the rate of convergence proved in refs. 9, 11, we recast this result in a contractive estimate between any two solutions with the implications discussed above. Moreover, this contraction property can be considered as the first Liapunov functional for the equilibrium solution of (1.3) and it gives not only convergence towards equilibrium but a measure on how any two solutions will converge to it and then a global trend of the system.[14]

Finally, we show that the convergence of Fourier metrics imply explicit rates of convergence of other probability metrics, in particular the Euclidean Wasserstein distance or Tanaka functional. These results give for the first time rates of convergence in the "physical space." On the other hand, we will discuss why in this case we cannot accomplish the whole argument done in ref. 4 due to the lack of propagation of smoothness. A result in this direction coupled with the results in this work implies by interpolation inequalities[4,13] an explicit rate of convergence in strong sense in Sobolev spaces. Fine overpopulated tail behavior properties of the self-similar profile will be dealt with in Sec. 5.

2. HOMOGENEOUS COOLING STATES: SELF-SIMILAR VARIABLES AND MOMENT EVOLUTION

We will start our analysis with a brief resumé of some questions related to the existence and uniqueness of homogeneous cooling states (HCS) for the inelastic Maxwell model (1.1). Let us remind that a homogeneous cooling state is a solution to (1.1) of the form:

$$f_{hc}(v,t) = \rho\theta_{hc}^{-\frac{3}{2}}(t) g_\infty\big((v-u)\theta_{hc}^{-\frac{1}{2}}(t)\big) \tag{2.1}$$

where $\theta_{hc}(t)$ is the second moment of f_{hc}, that decays following the law of cooling of the temperature for the inelastic Maxwell model (1.1) (see ref. 7):

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1-e^2}{4} B\theta^{\frac{3}{2}}. \tag{2.2}$$

Here, ρ and u are the density and mean velocity of the HCS, that, as briefly discussed in the introduction, are preserved in time and are fixed to be $\rho = 1$ and $u = 0$ for the rest of the paper.

Scaling the temporal variable as

$$\tau = \frac{B}{E}\int_0^t \theta^{\frac{1}{2}}(w)\,dw, \tag{2.3}$$

where $E = 8/(1-e^2)$, reduces Eq. (1.1) to

$$\frac{\partial f}{\partial \tau} = E Q(f, f). \tag{2.4}$$

This new time-scale makes the evolution law of the temperature for (2.4) to read as

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -2\theta\ . \tag{2.5}$$

Now, we scale with the own temperature of solutions in such a way that the homogeneous cooling states change into stationary states in the new variables. This is achieved through the change of variables (2.1):

$$f(v,\tau) = \theta^{-\frac{3}{2}}(\tau) g\big(v\theta^{-\frac{1}{2}}(\tau), \tau\big). \tag{2.6}$$

Substituting expression (2.6) into the Boltzmann equation, and supposing the solution to the Boltzmann equation has enough regularity, one obtains the equation satisfied by $g(\tilde{v}, \tau)$,

$$\frac{\partial g}{\partial \tau} + \nabla_{\tilde{v}} \cdot (\tilde{v} g(\tilde{v})) = \frac{E}{4\pi}\int_{\mathbb{R}^3}\int_{S^2} [\chi g(\tilde{v}^*) g(\tilde{w}^*) - g(\tilde{v}) g(\tilde{w})]\, dn\, d\tilde{w}. \tag{2.7}$$

As before, it is convenient to write Eq. (2.7) in weak form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbb{R}^3}\varphi(v)g(v,\tau)\,dv = \int_{\mathbb{R}^3} g(v)v\cdot\nabla\varphi(v)\,dv + E(\varphi, Q(g,g)). \tag{2.8}$$

By virtue of the self-similar change of variables (2.6), the temperature corresponding to the distribution $g(v,\tau)$ turns out to be equal to one independently of time, which implies that solutions to Eq. (2.8) are normalized with unit mass, zero mean velocity and unit temperature. Moreover, HCS for the original Eq. (2.4) are transformed into stationary solutions g_∞ of (2.8) with unit mass, zero mean velocity and unit temperature.

The Fourier transformed equation[6,7] corresponding to (2.4) is

$$\frac{\partial \hat f}{\partial \tau} = \frac{E}{4\pi}\int_{S^2}\left\{\hat f(k^+,\tau)\hat f(k^-,\tau) - \hat f(0,\tau)\hat f(k,\tau)\right\} dn \tag{2.9}$$

where

$$\begin{aligned} k^- &= \frac{1+e}{4}(k-|k|n),\\ k^+ &= \frac{3-e}{4}k + \frac{1+e}{4}|k|n. \end{aligned} \tag{2.10}$$

Likewise, the Fourier transformed equation corresponding to (2.7) reads

$$\frac{\partial \hat g}{\partial \tau} - (k\cdot\nabla_k)\,\hat g = E\left[\frac{1}{4\pi}\int_{S^2}\hat g(k^+)\hat g(k^-)\,dn - \hat g\right] = E\left[Q_+(\hat g,\hat g) - \hat g\right]. \tag{2.11}$$

The solution to (2.11) can be written in terms of the characteristics of the first order linear operator as

$$\hat g\left(\tau, k\mathrm{e}^{-\tau}\right) = \mathrm{e}^{-E\tau}\hat g(0,k) + E\int_0^\tau \mathrm{e}^{-E(\tau-s)}Q_+(\hat g,\hat g)\left(s, k\mathrm{e}^{-s}\right) ds. \tag{2.12}$$

Setting $\varphi(v) = v_i v_j$, $i \neq j$ into (2.8) shows that the evolution of the second cross moments for equation (2.8) is given by

$$\frac{\partial}{\partial \tau}\int_{\mathbb{R}^3} g(v)v_i v_j\,dv = -\frac{1+e}{1-e}\int_{\mathbb{R}^3} g(v)v_i v_j\,dv. \tag{2.13}$$

As a consequence, the non-isotropic part of the pressure tensor of the solutions vanishes if initially does so.

Due to its importance in the forthcoming analysis, we need also to study the propagation in time of higher moments of the solution to Eq. (2.8). To simplify notations, in what follows, for any $r \in \mathbb{N}$, we consider

$$M_{2r}(g(\tau)) = \int_{\mathbb{R}^3} g(v,\tau)|v|^{2r}\,dv,$$

and we will write $M_{2r}(g(\tau)) \equiv M_{2r}(\tau)$ whenever there is no confusion about the solution we discuss. We skip the details of the proof of next Lemma that are based on [Ref. 4, Lemma 2.3] and simple computations with the anti-drift term.

Lemma 2.1. (Time-dependent moment estimates) *Let $g(v, \tau)$ be the solution to equation (2.8), where the initial distribution $g_0(v)$ is such that $M_{2r}(g_0) < +\infty$ for some $r > 1$. Then, $M_{2r}(\tau)$ satisfies the following differential inequality*

$$\frac{d}{d\tau} M_{2r}(\tau) \leq -E\left[\frac{1-e^{2r}}{4}\big(M_{2r}(\tau) + M_{2(r-1)}(\tau)M_2(\tau)\big) - \frac{1}{2}\sum_{m=1}^{r-1}\binom{r}{m} M_{2(r-m)}(\tau)M_{2m}(\tau)\right] + 2r M_{2r}(\tau). \quad (2.14)$$

Consequently, $M_{2r}(\tau) < \infty$, for all $\tau > 0$, and bounded in $[0, T]$, for all $T > 0$.

3. HOMOGENEOUS COOLING STATES: EXISTENCE AND UNIQUENESS VIA CONTRACTION

We are now in a position to show the contraction of the d_s-distances for solutions to the scaled Eq. (2.8). The main consequence of this result will be the existence of a unique stationary state to Eq. (2.8) with unit mass, zero mean velocity and unit pressure tensor. To simplify computations, the proof that follows will be restricted to initial data g_0 with unit mass, zero mean velocity and unit pressure tensor, i.e.,

$$\int_{\mathbb{R}^3} g_0(v) v_i v_j dv = \delta_{ij}.$$

We remark that the proof of the first part of the following Theorem can be obtained by rephrasing arguments from [Ref. 9, Sec. 6] and [Ref. 11, Sec. 4].

Theorem 3.1. (Strict contraction for the scaled equation) *Let $\hat{g}_1$ and $\hat{g}_2$ be two solutions to (2.11) corresponding to initial values $\hat{g}_1(0), \hat{g}_2(0)$ with unit mass, zero mean velocity and unit pressure tensor, i.e.,*

$$\int_{\mathbb{R}^3} g_1(v) v_i v_j dv = \int_{\mathbb{R}^3} g_2(v) v_i v_j dv = \delta_{ij}. \quad (3.1)$$

Then $d_{2+\alpha}(\hat{g}_1(0), \hat{g}_2(0)) < \infty$, $0 < \alpha < 1$, and there exists an explicit constant $C(\alpha, e) > 0$, $C(\alpha, e) \to 0$ as $\alpha \to 0$, such that

$$d_{2+\alpha}(\hat{g}_1(\tau), \hat{g}_2(\tau)) \leq d_{2+\alpha}(\hat{g}_1(0), \hat{g}_2(0)) \mathrm{e}^{-C(\alpha,e)\tau}, \quad (3.2)$$

for any $\tau \geq 0$. Consequently, Eq. (2.11) has a unique steady state $\hat{g}_\infty$ which belongs to the set of probability measures with unit mass, zero mean velocity and pressure tensor given by (3.1).

Proof. We remark first that equality of all moments up to order 2 implies that the distance $d_{2+\alpha}$, $0 < \alpha < 1$, between g_1 and g_2 is well-defined.

Step 1: Estimates on the distance. From Eq. (2.12) it follows:

$$\mathrm{e}^{E\tau}\frac{(\hat{g}_1-\hat{g}_2)(\tau,k\mathrm{e}^{-\tau})}{|k|^{2+\alpha}} = \mathrm{e}^{(E-(2+\alpha))\tau}\frac{(\hat{g}_1-\hat{g}_2)(\tau,k\mathrm{e}^{-\tau})}{|k\mathrm{e}^{-\tau}|^{2+\alpha}} = \frac{\hat{g}_1(0,k)-\hat{g}_2(0,k)}{|k|^{2+\alpha}}$$
$$+E\int_0^\tau \mathrm{e}^{(E-(2+\alpha))s}\frac{(Q_+(\hat{g}_1,\hat{g}_1)-Q_+(\hat{g}_2,\hat{g}_2))\left(s,k\mathrm{e}^{-s}\right)}{|k\mathrm{e}^{-s}|^{2+\alpha}}\,ds. \tag{3.3}$$

As shown in ref. 4,

$$\left|\frac{(Q_+(\hat{g}_1,\hat{g}_1)-Q_+(\hat{g}_2,\hat{g}_2))\,(k)}{|k|^{2+\alpha}}\right| = \frac{1}{4\pi}\left|\int_{S^2}\frac{\hat{g}_1(k^+)\hat{g}_1(k^-)-\hat{g}_2(k^+)\hat{g}_2(k^-)}{|k|^{2+\alpha}}dn\right|$$
$$\leq A(\alpha,e)\sup_{k\in\mathbb{R}^3}\frac{|\hat{g}_1(k)-\hat{g}_2(k)|}{|k|^{2+\alpha}} \tag{3.4}$$

where $A(\alpha, e)$ is given by

$$A(\alpha,e) = \frac{1}{4\pi}\int_{S^2}\frac{|k^+|^{2+\alpha}+|k^-|^{2+\alpha}}{|k|^{2+\alpha}}\,dn \tag{3.5}$$

and $A(\alpha, e) \leq A(0, e) = (e^2 + 3)/4 < 1$ for each restitution coefficient $e \neq 1$ (see ref. 4). In fact, it can be checked by inserting the expressions of k^- and k^+ (see (2.10)) into (3.5) that

$$A(\alpha,e) = \frac{1}{2}\int_0^\pi\left\{\left[\left(\frac{1+e}{4}\right)^2 2(1-\cos\theta)\right]^{\frac{2+\alpha}{2}}\right.$$
$$\left.+\left[\left(\frac{3-e}{4}\right)^2+\left(\frac{1+e}{4}\right)^2+2\left(\frac{3-e}{4}\right)\left(\frac{1+e}{4}\right)\cos\theta\right]^{\frac{2+\alpha}{2}}\right\}\sin\theta\,d\theta$$
$$= \frac{2}{4+\alpha}\left[\left(\frac{1+e}{2}\right)^{2+\alpha} + \frac{1-\left(\frac{1-e}{2}\right)^{4+\alpha}}{1-\left(\frac{1-e}{2}\right)^2}\right].$$

Hence, taking the supremum of (3.3) we obtain

$$\mathrm{e}^{(E-(2+\alpha))\tau}d_{2+\alpha}(\hat{g}_1,\hat{g}_2)(\tau) \leq d_{2+\alpha}(\hat{g}_1(0),\hat{g}_2(0))$$
$$+A(\alpha,e)E\int_0^\tau \mathrm{e}^{(E-(2+\alpha))s}d_{2+\alpha}(\hat{g}_1,\hat{g}_2)(s)\,ds.$$

Let us set $w(\tau) = \mathrm{e}^{(E-(2+\alpha))\tau} d_{2+\alpha}(\hat{g}_1, \hat{g}_2)(\tau)$. Then

$$w(\tau) \leq w(0) + A(\alpha, e)E \int_0^{\tau} w(s)\, ds, \tag{3.6}$$

which, by Gronwall inequality, implies $w(\tau) \leq w(0)\mathrm{e}^{A(\alpha,e)E\tau}$. Hence

$$d_{2+\alpha}(\hat{g}_1, \hat{g}_2) \leq d_{2+\alpha}(\hat{g}_1(0), \hat{g}_2(0))\mathrm{e}^{-C(\alpha,e)\tau}, \tag{3.7}$$

with

$$C(\alpha, e) = E(1 - A(\alpha, e)) - (2 + \alpha) = E(1 - G(\alpha, e)) \tag{3.8}$$

where $G(\alpha, e) = A(\alpha, e) + \frac{1-e^2}{8}(2 + \alpha)$.

Step 2: Strict contraction of the distance. The first part of the theorem will follow if we show that for all $0 < \alpha \leq 1$ and all restitution coefficients $0 \leq e < 1$, $G(\alpha, e) < 1$.

We leave the details to the reader and sketch the arguments. We first remark that for any given e, the function $G(\alpha, e)$ is convex with respect to the variable α by a direct computation of the second derivative. Now, let us point out that $G(0, e) = A(0, e) + \frac{1-e^2}{4} = 1$ for all e. Moreover, a detailed analysis of the behavior of the function $G(1, e)$ shows that $G(1, e) < 1$ for all e.

From this information, we can conclude that for all α such that $0 < \alpha \leq 1$, it holds $G(\alpha, e) < 1$ in the whole interval $0 \leq e < 1$, and thus, $C(\alpha, e)$ is strictly positive for all $0 < \alpha \leq 1$ and $0 \leq e < 1$.

Step 3: Existence and uniqueness of steady state. The final part of this theorem follows the same lines as [Ref. 4, Theorem 3.2].

Take the set X_α defined as the subset of $\mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ with moments up to order 2 given by (3.1). This set is a complete metric space endowed with the distance $d_{2+\alpha}$ being a closed subset of $\mathcal{P}_{2+\alpha}(\mathbb{R}^3)$, see [Ref. 21, Theorem 1]. Let us consider the flow map of (2.11),

$$T_\tau : (X_\alpha, d_{2+\alpha}) \longrightarrow (X_\alpha, d_{2+\alpha}),$$

for any time $\tau > 0$, given by $T_\tau(g_0) = g(\tau)$ with $g(\tau)$ the unique solution at time τ of (2.11) with initial datum $g_0 \in X_\alpha$.

The first and second steps prove that T_τ is a uniform contraction from the complete metric space $(X\hat{\alpha}, d_{2+\alpha})$ into itself with contraction constant $\mathrm{e}^{-C(\alpha,e)\tau} < 1$. Therefore, Banach fixed point theorem ensures the existence and uniqueness of a fixed point $g_\infty(\tau)$ for T_τ in $(X\hat{\alpha}, d_{2+\alpha})$. A simple argument using the continuity in time of the nonlinear semigroup solution to (2.11) implies that these unique fixed points cannot depend on time, $g_\infty(\tau) = g_\infty$, for all $\tau \geq 0$, and the existence of a unique fixed point follows. It remains to show that the unique fixed point $g_\infty \in X_\alpha$ is a stationary solution to (2.11). We use the argument of ref. 19. By (3.4), for all

$g \in X_\alpha$ we have

$$d_{2+\alpha}(Q_+(\hat{g}(\tau), \hat{g}(\tau)), Q_+(\hat{g}_\infty, \hat{g}_\infty)) \leq A(\alpha, e) d_{2+\alpha}(\hat{g}(\tau), \hat{g}_\infty).$$

This implies the weak* convergence of $Q_+(g(\tau), g(\tau))$ towards $Q_+(g_\infty, g_\infty)$. In particular, due to the equivalence among different metrics which metrize the weak* convergence of measures,[17,21] if $C_0^1(\mathbb{R}^3)$ denotes the set of compactly supported continuously differentiable functions, endowed with its natural norm $||\cdot||_1$, for all $\varphi \in C_0^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \varphi(v) Q_+(g(\tau), g(\tau))(v)\, dv \to \int_{\mathbb{R}^3} \varphi(v) Q_+(g_\infty, g_\infty)(v)\, dv.$$

On the other hand, for all $\varphi \in C_0^1(\mathbb{R}^3)$, since $|v \cdot \nabla\varphi(v)| \leq |v| \|\nabla\varphi\|_1$, and the second moment of $g(v, \tau)$ is equal to unity, the convergence of $d_{2+\alpha}(g(\tau), g_\infty)$ to zero implies

$$\int_{\mathbb{R}^3} v \cdot \nabla\varphi(v) g(v, \tau)\, dv \to \int_{\mathbb{R}^3} v \cdot \nabla\varphi(v) g_\infty(v)\, dv.$$

Finally, for all $\varphi \in C_0^1(\mathbb{R}^3)$ it holds

$$\int_{\mathbb{R}^3} g_\infty(v) v \cdot \nabla\varphi(v)\, dv + E(\varphi, Q(g_\infty, g_\infty)) = 0.$$

This shows that g_∞ is a stationary solution to (2.8). □

Remark 3.2. (Comparisons to Previous Results). *Let us repeat those parts of Step 1 and 2 in previous theorem reformulate results in [Ref. 9, Sec. 6] and [Ref. 11, Sec. 4] with our point of view. The main novelties are:*

1. *To recast the result as a strict contractivity of the distance $d_{2+\alpha}$ along the flow. More general contraction properties will be shown in next section.*
2. *The existence and uniqueness of the steady state, and thus of the HCS, follow in a very elegant and straightforward way avoiding the eigenvalue analysis of an integral operator related to Q_+, used for the existence proof of steady state in ref. 9. Nevertheless, they are intimately related since the spectral gap of the linearized operator in ref. 9 is exactly the contractivity constant in (3.2). Furthermore, our result can be considered as an easy proof of a spectral gap estimate for the linearized operator.*
3. *Decay rates towards the steady state are then easily obtained by specializing one of the two solutions as the steady state. However, the information of our theorem is much stronger since it gives the nonlinear stability of the steady state and it measures the control between any two solutions. In*

fact, (3.6) in previous theorem can be recast into

$$\frac{d^+}{d\tau} d_{2+\alpha}(\hat{g}(\tau), \hat{g}_\infty) \leq -C(\alpha, e) d_{2+\alpha}(\hat{g}(\tau), \hat{g}_\infty)$$

for all $\tau > 0$ and all solutions corresponding to initial data in Theorem 3.1. Therefore, $d_{2+\alpha}(\hat{g}, \hat{g}_\infty)$ is the first known Liapunov functional, cf. ref. 4, to our knowledge, for non-trivial steady states of inelastic Maxwell models. Decay rates for more general initial data will be reviewed in next section.

Remark 3.3. (Related Works and Open Problems) *Let us finally mention that similar ideas were used in nonlinear diffusion and nonlinear friction equations before in ref. 14 with the Euclidean Wasserstein distance. It is an open problem to prove or disprove that the Euclidean Wasserstein distance, also called the Tanaka functional by the kinetic community, is a strict contraction for this inelastic Maxwell model. In the elastic case it is known to be a non-expansive contraction.*[20]

As a simple corollary coming back to original variables using the time and spatial changes of variables (2.3) and (2.6), we obtain the existence and uniqueness of homogeneous cooling states (HCS) for the original Eq. (2.4).

Corollary 3.4. (Existence-Uniqueness of HCS) *Eq. (1.1) has a unique homogeneous cooling state with unit mass and zero mean velocity given by*

$$f_{hc}(t) = \theta_{hc}^{-\frac{3}{2}}(t) g_\infty\left(\theta_{hc}^{-\frac{1}{2}}(t) v\right)$$

where the temperature $\theta_{hc}(t)$ is given by (2.2) fixing any initial value $\theta_{hc}(0) > 0$. Moreover, all homogeneous cooling states of (1.1) are given in terms of g_∞ by means of (2.1).

4. HOMOGENEOUS COOLING STATES AND THE ERNST-BRITO CONJECTURE

Few years ago Ernst and Brito,[15,16] by combining results on scaling solutions and overpopulated high energy tails in inelastic hard sphere fluids and inelastic Maxwell models with an old conjecture of Krook and Wu[5,18] on a special self-similar solution of the elastic Boltzmann equation, named Bobylev, Krook and Wu (BKW) mode, formulated a conjecture on the role of the homogeneous cooling states. This conjecture reads:

(EBC1) The HCS should be attractors for large sets of initial data for large times (see [Ref. 16, Sec. 4]).

(EBC2) The HCS should have overpopulated high energy tails. Hence, moments of the HCS, $M_{2r}(g_\infty)$, are bounded if and only if $r < r_{EB}(e)$ where $r_{EB}(e)$ is characterized by the unique solution to the equation

$$\frac{1-e^2}{4}r = 1 - A(2r-2,e) = 1 - \frac{1}{1+r}\left[\left(\frac{1+e}{2}\right)^{2r} + \frac{1-\left(\frac{1-e}{2}\right)^{2r+2}}{1-\left(\frac{1-e}{2}\right)^2}\right]. \tag{4.1}$$

This equation (see [Ref. 16, Eq. (3.13)]) for capturing the high energy tails of the distribution function was also obtained in refs. 2, 3.

Later, Bobylev and Cercignani[9] and Bobylev, Cercignani and Toscani[11] proved the first part of the Ernst-Brito conjecture **(EBC1)**. More precisely, they proved the convergence weakly as measures of any solution with finite energy to (1.1) towards the HCS with the same initial density and mean velocity by assuming the additional hypothesis that a moment of order $2+\epsilon$ is initially finite. We will further improve the understanding of **(EBC1)** in Subsections 4.2 and 4.3 by reckoning an explicit decay rate towards HCS based on the contraction of the metrics. This will imply that the self-similar profile is approached by general initial conditions at an algebraic rate in time with explicit constants.

Concerning the second part of the Ernst-Brito conjecture **(EBC2)**, Bobylev and Cercignani[9] showed that this result is essentially true except for a finite number of restitution coefficients close to the elastic limit, $e = 1$, in which they have more moments finite than the ones implied by (4.1). More precisely, they claimed that if $r < r_{EB}(e)$, the moments of order $2r$ are finite and the converse is true for "almost all" coefficients of restitution, see [Ref. 9, Lemma 8.1, Theorem 7.2] for a detailed description of the result. Recently, these authors together with Gamba[10,12] have found a gap in the argument leading to this exceptional set of restitution coefficients and have corrected this result proving that the moments of order $2r$ of the HCS are finite if and only if $r < r_{EB}(e)$.

Here in the next subsection, we will show using the contraction result that all moments of the HCS similarity solution are bounded for $r < r_{EB}(e)$ and moreover this implies a uniform in time control of suitable moments of the solutions of the Cauchy problem.

4.1. Tails of the HCS and Moments of Solutions: Second Part of EB Conjecture

We remark that to look for solutions to Eq. (4.1) that gives the optimal exponent $r_{EB}(e)$ for bounded moments of the similarity solution is equivalent to look for the value of α_e such that $C(\alpha_e, e) = 0$. Indeed, $2r_{EB}(e) = 2 + \alpha_e$. Therefore, taking into account our contraction result, Theorem 3.1, and the second part of the Ernst-Brito conjecture **(EBC2)** proved in ref. 9 and corrected in refs. 10, 12, we can draw the following conclusion proved in the next result for any fixed value of the coefficient of restitution e: the flow of equation (1.1) is strictly contractive in $d_{2+\alpha}$ if and only if the moments of order $2+\alpha$ of the similarity solution g_∞ are bounded.

Theorem 4.1. (Boundedness of moments for the HCS.) *If the condition*

$$G(\alpha, e) < 1 \Longleftrightarrow C(\alpha, e) > 0 \Longleftrightarrow 2+\alpha < 2r_{EB}(e)$$

holds, then the distance $d_{2+\alpha}$ is strictly contractive for initial data having equal moments up to order $2+[\alpha]$, for $\alpha \notin \mathbb{N}$, and $1+\alpha$ equal moments and bounded $2+\alpha$ moments whenever $\alpha \in \mathbb{N}$. Furthermore, this fact implies that the moments of order $2+\alpha$ of the homogeneous cooling state g_∞ are bounded.

Proof. This theorem follows the same lines as [Ref. 4, Theorem 3.2]. Thanks to Lemma 2.1, moments are propagated, i.e., the $2+\alpha$-th isotropic moment is bounded, not uniformly in time, if the same moment is bounded initially.

Step 1: Case $\alpha \in \mathbb{N}$. We can repeat the same argument of Theorem 3.1, by computing now the evolution of the distance with index $2+\alpha$ between any two solutions allowing $\alpha \geq 1$. One proves the following assertion: given any natural $\alpha \geq 1$, let $\hat{g}_1(0)$ and $\hat{g}_2(0)$ be two initial data to (2.11) with equal moments up to order $1+\alpha$ and finite moments of order $2+\alpha$, then $d_{2+\alpha}(\hat{g}_1(0), \hat{g}_2(0)) < \infty$ and

$$d_{2+\alpha}(\hat{g}_1(\tau), \hat{g}_2(\tau)) \leq d_{2+\alpha}(\hat{g}_1(0), \hat{g}_2(0)) \mathrm{e}^{-C(\alpha,e)\tau}, \tag{4.2}$$

for any $\tau \geq 0$.

Let us point out that (4.2) implies that moments of the solutions remain equal up to order $1+\alpha$, since initially they are equal, and thus, the distance $d_{2+\alpha}(\hat{g}_1(\tau), \hat{g}_2(\tau)) < \infty$, for all $\tau \geq 0$. Moreover, the contraction is strict if $C(\alpha, e) > 0$.

Now, we proceed by induction on α. We already know that the steady state g_∞ has bounded second moments. Let us assume that $\alpha \geq 1$ and that g_∞ has moments bounded up to order $1+\alpha$. Take the set X_α defined as the subset of $\mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ with equal moments to those of g_∞ up to order $1+\alpha$. This set is a complete metric space endowed with the distance $d_{2+\alpha}$ being a closed subset of $\mathcal{P}_{2+\alpha}(\mathbb{R}^3)$. Proceeding as in the proof of Theorem 3.1, the flow map T_τ is a uniform contraction from $(X_\alpha, d_{2+\alpha})$ into itself with contraction constant less than 1 whenever $C(\alpha, e)$ is positive. Therefore, T_τ has a unique steady state $g \in X_\alpha$ and thus, by uniqueness of steady state in X_β, for $0 < \beta < 1$, we conclude $g = g_\infty \in (X_\alpha, d_{2+\alpha})$ and thus, g_∞ has finite moments of order $2+\alpha$.

Step 2: Case $\alpha \notin \mathbb{N}$. In the present case the decay (4.2) holds if the initial data $\hat{g}_1(0)$ and $\hat{g}_2(0)$ have equal moments up to order $2+[\alpha]$. Thanks to Step 1, g_∞

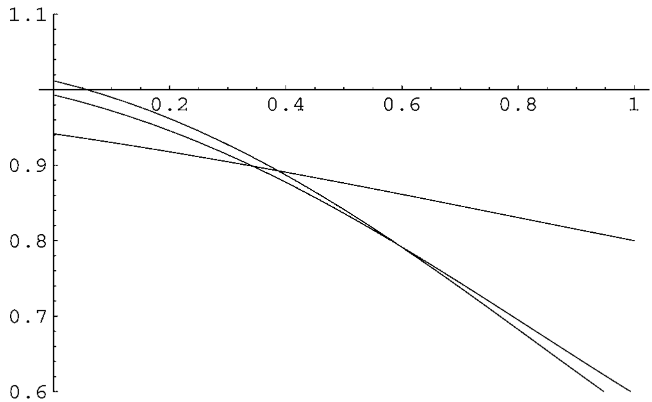

Fig. 1. The functions $G(\alpha, e)$ for values $\alpha = 1, \alpha = 2.7$ and $\alpha = 3$.

has bounded moments of order $2 + [\alpha]$. Take now the set X_α defined as the subset of $\mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ with equal moments to those of g_∞ up to order $2 + [\alpha]$. Analogously to Step 1, it can be shown that the flow map T_τ has a unique steady state in X_α, and consequently g_∞ has finite moments of order $2 + \alpha$. □

In Figures 4 and 2, we show the largest root α_e of $C(\alpha, e) = 0$ in terms of e, which corresponds to compute $r_{EB}(e) = 1 + \frac{\alpha_e}{2}$. In fact, taking into account [Ref. 9, Theorem 7.2] and refs. 10, 12, we obtain the following corollary.

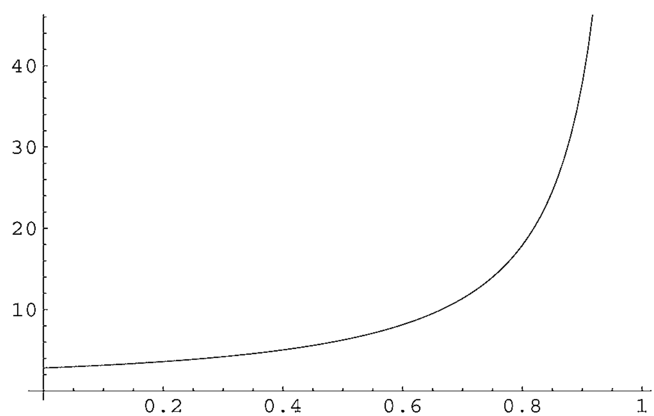

Fig. 2. The largest zero α_e of $C(\alpha, e) = 0 \Longleftrightarrow G(\alpha, e) = 1$ as a function of e, for instance $\alpha_0 = 2.81307$, and thus, $r_{EB}(0) = 2.40653$ by Newton-Raphson method.

Corollary 4.2. (Optimality of the contraction result) *The flow map for equation (1.1) is a strict contraction for the distance $d_{2+\alpha}$ if and only if*

$$G(\alpha, e) < 1 \Longleftrightarrow C(\alpha, e) > 0 \Longleftrightarrow 2 + \alpha < 2r_{EB}(e),$$

or equivalently if and only if the moments of order $2 + \alpha$ of the homogeneous cooling state g_∞ are bounded.

We will now make use of these new contractive bounds to estimate uniformly the moments of solutions of the Cauchy problem for certain initial data. This is based on the observation that distances d_m with $m \in \mathbb{N}, m \geq 2$, control the difference of the moments of order m.

Theorem 4.3. (Uniform in time moment estimates.) *Given an even $m \in \mathbb{N}$ with $2 < m < 2r_{EB}(e)$. Let $g(v, \tau)$ be the solution to equation (2.8) for an initial datum $g_0(v)$ having equal moments to those of g_∞ up to order $m - 1$ and bounded m-moments. Then the moments of order m of $g(v, \tau)$ are uniformly bounded in time. In particular, given an initial datum g_0 with equal moments to those of g_∞ up to third order, and fourth-order moment bounded, then the fourth moments of the solution are uniformly bounded in time for all e.*

Proof. We start with the following result extracted from similar arguments in ref. 24. Let us fix $\eta \in \mathbb{R}^3$ with $|\eta| = 1$ and $\lambda \geq 0$. Let us denote by $D^m \hat{f}$ the differential tensor of order m of $\hat{f}$. Given two distributions g_1, g_2 with moments of order m bounded and equal moments up to order $m - 1$, their Fourier transform are m continuously differentiable and

$$D^m(\hat{g}_1 - \hat{g}_2)(0)(\eta, \cdots, \eta) = \lim_{\lambda \to 0^+} \frac{\hat{g}_1(\lambda\eta) - \hat{g}_2(\lambda\eta)}{\lambda^m}.$$

Now, putting this together with the definition of d_m, we get

$$|D^m(\hat{g}_1 - \hat{g}_2)(0)(\eta, \cdots, \eta)| \leq d_m(\hat{g}_1, \hat{g}_2)$$

for all $\eta \in \mathbb{R}^3$ with $|\eta| = 1$. Therefore, we have shown that given g_1, g_2 with moments of order m bounded and equal moments up to order $m - 1$, we have

$$\left| \int_{\mathbb{R}^3} |v|^m (g_1 - g_2)\, dv \right| \leq C_m d_m(\hat{g}_1, \hat{g}_2). \tag{4.3}$$

Applying previous estimate with $g_1 = g(\tau)$ and $g_2 = g_\infty$, we deduce

$$\left| \int_{\mathbb{R}^3} |v|^m (g(v, \tau) - g_\infty(v))\, dv \right| \leq C_m d_m(\hat{g}(\tau), \hat{g}_\infty),$$

that together with (4.2), implies

$$\left| \int_{\mathbb{R}^3} |v|^m (g(v,\tau) - g_\infty(v))\, dv \right| \le C_m d_m(\hat{g}(0), \hat{g}_\infty) \mathrm{e}^{-C(\alpha,e)\tau}.$$

Finally, we conclude by using that the moment of order m of g_∞ is bounded for $2 < m < 2r_{EB}(e)$. Since $r_{EB}(e) > 2$ for all e, we obtain the particular statement, i.e., the uniform control on the fourth moment of solutions of the Cauchy problem for all e. □

It is an open problem to get uniform in time estimates of the moments of solutions for more general initial data.

4.2. Explicit Decay Rates Towards Self-Similarity: First Part of the EB Conjecture

In this subsection, we plan to get rid of the assumption of equal second moments in Theorem 3.1, in order to prove the exponential convergence of each solution $f(v,\tau)$ of Eq. (2.4) corresponding to a general initial datum, towards the corresponding similarity solution $f_{hc}(\tau)$ in d_2. Let us remark that neither Theorem 3.1 nor the results contained in ref. 11 give any decay rate in the case of d_2. In fact, Eq. (2.8) is a non-strict contraction for d_2, i.e.,

$$d_2(\hat{g}_1(\tau), \hat{g}_2(\tau)) \le d_2(\hat{g}_1(0), \hat{g}_2(0)) \tag{4.4}$$

for any $\tau \ge 0$ and any $\hat{g}_1, \hat{g}_2$ solutions to (2.11) corresponding to initial data with unit mass, zero mean velocity and second moment bounded.

In order to do this, let us consider the evolution of the pressure tensor for the solutions $f(v,\tau)$ of Eq. (2.4). For $i \ne j$ the quantity

$$p_{ij}(\tau) = \int_{\mathbb{R}^3} v_i v_j f(v,\tau)\, dv$$

is governed by the equation

$$\frac{dp_{ij}}{d\tau} = -\frac{(1+e)(3-e)}{8} E p_{ij}. \tag{4.5}$$

If $\hat{\Phi}(k,\tau)$ is defined as

$$\hat{\Phi}(k,\tau) = \begin{cases} -\frac{1}{2} \sum\limits_{i \ne j} p_{ij}(\tau) k_i k_j & if \quad |k| \le 1 \\ 0 & if \quad |k| > 1 \end{cases}, \tag{4.6}$$

we will show that the contraction in $d_{2+\alpha}$ of the non-isotropic part $\hat{f}(\tau) - \hat{\Phi}(\tau)$ together with the decay of the pressure tensor of the solution towards the pressure tensor of the HCS $\hat{f}_{hc}$ is enough to ensure the convergence of the solution towards the HCS in d_2. In the proof we shall resort to the contraction in $d_{2+\alpha}, \alpha > 0$, and thus, we need an additional assumption on the initial data, i.e., to have the corresponding moment of order $2 + \alpha$ finite. The following theorem can be also obtained from, [Ref. 11, Sec. 5] but here we recast it as an estimate in d_2 and we work out the constants explicitly. We include the proof in the Appendix.

Theorem 4.4. (Decay result for general initial data.) *Let $\hat{f}(k, \tau)$ be the solution of the time-scaled inelastic Maxwell equation (2.9) corresponding to the initial datum $\hat{f}(0)$ with unit mass, zero mean velocity such that $d_{2+\alpha}(\hat{f}(0) - \hat{\Phi}(0), \hat{f}_{hc}(0)) < \infty$, where $\hat{f}_{hc}$ denotes the corresponding self-similar solution. Then there exists $C_1 > 0$ such that*

$$d_{2+\alpha}(\hat{f}(\tau) - \hat{\Phi}(\tau), \hat{f}_{hc}(\tau)) \leq \left[2d_{2+\alpha}(\hat{f}(0) - \hat{\Phi}(0), \hat{f}_{hc}(0)) + C_1\right] \mathrm{e}^{-(1-A(\alpha,e))E\tau} \tag{4.7}$$

for any $0 < \alpha < 1$.

Next Lemma will be useful to relate different metrics.

Lemma 4.5. (Interpolation of metrics.) *Let $q > p$, $|\hat{f}| \leq 1$, $|\hat{g}| \leq 1$, then*

$$d_p(\hat{f}, \hat{g}) \leq 2\left(\frac{q-p}{2p}\right)^{p/q} \frac{q}{q-p}\left[d_q(\hat{f}, \hat{g})\right]^{p/q} = C_{p,q}\left[d_q(\hat{f}, \hat{g})\right]^{p/q}. \tag{4.8}$$

Proof. Since $q > p$, for any $R > 0$ it holds

$$\begin{aligned} d_p(\hat{f}, \hat{g}) &= \sup_{k \in \mathbb{R}^3} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^p} \leq \sup_{|k| \leq R} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^p} + \sup_{|k| > R} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^p} \\ &\leq \sup_{|k| \leq R} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^q} R^{q-p} + \frac{2}{R^p} \leq d_q(\hat{f}, \hat{g}) R^{q-p} + \frac{2}{R^p}. \end{aligned} \tag{4.9}$$

Optimizing the function $y(R) = AR^{q-p} + \frac{2}{R^p}$ (with $A > 0$) over R concludes. □

From Theorem 4.4 and Lemma 4.5 it follows the exponential decay of the distance $d_2(\hat{f}(\tau), \hat{f}_{hc}(\tau))$.

Theorem 4.6. (Decay rate towards self-similarity for general initial data.) *Let $\hat{f}(k, \tau)$ be the solution of the time-scaled inelastic Maxwell equation (2.9) corresponding to the initial datum $\hat{f}(0)$ with unit mass, zero mean velocity and second moment bounded. Then there exist explicit constants $C_1, C_2 > 0$, depending on second moments of the initial data, such that*

$$d_2(\hat{f}(\tau), \hat{f}_{hc}(\tau)) \leq C_{2,2+\alpha}\left\{\left[2d_{2+\alpha}(\hat{f}(0) - \hat{\Phi}(0), \hat{f}_{hc}(0)) + C_1\right]\right.$$

$$\left.\times \exp\{-(1-A(\alpha,e))E\tau\}\right\}^{\frac{2}{2+\alpha}} + C_2 \exp\left\{-\frac{3-e}{1-e}\tau\right\}. \tag{4.10}$$

Proof. The distance $d_2(\hat{f}(\tau), \hat{f}_{hc}(\tau))$ can be split as

$$d_2(\hat{f}(\tau), \hat{f}_{hc}(\tau)) \leq \sup_{k\in\mathbb{R}^3} \frac{|\hat{f}(k,\tau) - \hat{\Phi}(k,\tau) - \hat{f}_{hc}(k,\tau)|}{|k|^2} + \sup_{k\in\mathbb{R}^3} \frac{|\hat{\Phi}(k,\tau)|}{|k|^2}$$

$$= d_2(\hat{f}(\tau) - \hat{\Phi}(\tau), \hat{f}_{hc}(\tau)) + \sup_{|k|\leq 1} \frac{|\hat{\Phi}(k,\tau)|}{|k|^2}. \tag{4.11}$$

By applying Lemma 4.5 with $p = 2$ and $q = 2 + \alpha$,

$$d_2(\hat{f}(\tau) - \hat{\Phi}(\tau), \hat{f}_{hc}(\tau)) \leq C_{2,2+\alpha}\left[d_{2+\alpha}(\hat{f}(\tau) - \hat{\Phi}(\tau), \hat{f}_{hc}(\tau))\right]^{\frac{2}{2+\alpha}},$$

hence from Theorem 4.4 we get the first term in the right-hand side of (4.10). Owing to the definition of $\hat{\Phi}(k,\tau)$, the last term of (4.11) can be estimated by means of the law (4.5) which describes the evolution of the pressure tensor:

$$\sup_{|k|\leq 1} \frac{|\hat{\Phi}(k,\tau)|}{|k|^2} \leq \frac{1}{2}\left(\max_{i\neq j} |p_{ij}(\tau)|\right) \sup_{|k|\leq 1}\left\{\frac{1}{|k|^2}\sum_{i\neq j}|k_i k_j|\right\}$$

$$\leq \left(\max_{i\neq j}|p_{ij}(0)|\right)\exp\left\{-\frac{(1+e)(3-e)}{8}E\tau\right\}$$

and this concludes the proof. □

Remark 4.7. (Exponential decay result in scaled variables). *Given $\hat{g}$ a solution to (2.11) corresponding to the initial value $\hat{g}(0)$ with unit mass, zero mean velocity and bounded second moment, then*

$$d_2(\hat{g}(\tau), \hat{g}_\infty) \leq \frac{C_{2,2+\alpha}}{\theta_0}\left[2d_{2+\alpha}(\hat{g}(0) - \hat{\Phi}(0), \hat{g}_\infty) + C_1\right]^{2/(2+\alpha)}$$
$$\times \exp\left\{-\frac{2}{2+\alpha}C(\alpha,e)\tau\right\} + \frac{C_2}{\theta_0}\exp\left\{-\frac{1+e}{1-e}\tau\right\}. \tag{4.12}$$

This is a direct consequence of $\hat{g}(k,\tau) = \hat{f}(k\theta^{-\frac{1}{2}}(\tau),\tau)$, the scaling property of d_2, $d_2(\hat{g}, \hat{g}_\infty) = d_2(\hat{f}, \hat{f}_{hc})\theta^{-1}(\tau)$, and the evolution of the temperature (2.5), $\theta(\tau) = \theta_0 \mathrm{e}^{-2\tau}$.

Remark 4.8. (Algebraic decay result in original variables). The evolution equation (2.2) yields $\theta(t) = (\theta_o^{-\frac{1}{2}} + \frac{1-e^2}{8}Bt)^{-2}$, hence time scaling (2.3) is nothing but

$\tau = \log[1 + (B/(E\theta_0^{-1/2}))t]$. Therefore, to any exponential decay in the variable τ there corresponds an algebraic decay in t. For instance, from Theorem 4.6 we get the following estimate for the convergence of each solution $f(v, t)$ towards the homogeneous cooling state $f_{hc}(t)$:

$$d_2(\hat{f}(t), \hat{f}_{hc}(t)) \le C_{2,2+\alpha}\left[2d_{2+\alpha}(\hat{f}(0) - \hat{\Phi}(0), \hat{f}_{hc}(0)) + C_1\right]^{2/(2+\alpha)}$$
$$\times \left[1 + \frac{B}{E\theta_0^{-\frac{1}{2}}}t\right]^{-(2(1-A(\alpha,e))E)/(2+\alpha)} + C_2\left[1 + \frac{B}{E\theta_0^{-\frac{1}{2}}}t\right]^{-(3-e)/(1-e)}$$

4.3. Decay Rates in "Physical Space": Convergence in Euclidean Wasserstein Distance

A natural question on the decay rates is if one can extend the interpolation ideas to deduce decay rates in classical Sobolev spaces and in L^1, that is, directly in the physical space. These results were accomplished in ref. 4, for the heat bath case, based on the ideas of ref. 13. In fact, it is a general principle that assuming one has uniform in time control of moments and uniform in time propagation of smoothness, then once you have proved an exponential decay in d_2, you obtain the same decay with almost the same exponent in all Sobolev spaces and in L^1.

Summarizing, the proof of such a result requires the knowledge of the eventual propagation of moments and regularity for the solution to Eq. (2.8). We have shown the uniform in time propagation of moments, at least for particular initial data, in Subsection 4.1. Unlikely, it is not clear that the solution to Eq. (2.8) satisfies uniform in time regularity estimates.

Nevertheless, we can show using only the uniform in time bound of the fourth moment proven in Theorem 4.3, that the Euclidean Wasserstein distance between certain solutions to (2.11) and g_∞ converges exponentially to zero as $t \to \infty$.

Let us briefly remind that the Euclidean Wasserstein distance, also known as Tanaka functional[20] in the kinetic community, is defined as

$$W_2(g_0, g_1) = \inf\left\{\int_{\mathbb{R}^3\times\mathbb{R}^3} |v - w|^2 d\gamma(v, w); \quad \gamma \in \Gamma(g_0, g_1)\right\}^{1/2}; \qquad (4.13)$$

here $\Gamma(g_0, g_1)$ is the set of probability measures on $\mathbb{R}^3 \times \mathbb{R}^3$ having marginals g_0 and g_1. This metric is well defined in $\mathcal{P}_2(\mathbb{R}^3)$ and is equivalent to the weak* convergence as measures. In fact, it was proven in ref. 21, that both d_2 and W_2 give the same weak* uniformity in the set

$$\mathcal{S}_{\alpha,M} = \left\{g \in \mathcal{P}_2(\mathbb{R}^3) \text{ such that } \int_{\mathbb{R}^3} |v|^{2+\alpha} g(v)\, dv \le M\right\}$$

with α, $M > 0$. In order to explicitly write the decay rate in W_2, we need to review and introduce other probability metrics used in refs. 17, 21:

a) Prokhorov's distance $\Pi(g_0, g_1)$: for $\delta \geq 0$ and $U \subset \mathbb{R}^3$, we define

$$U^\delta = \{v \in \mathbb{R}^3; d(v, U) < \delta\}, \quad U^{\delta]} = \{v \in \mathbb{R}^3; d(v, U) \leq \delta\}$$

where $d(v, U) = \inf\{\|v - w\|, w \in U\}$. Let

$$\sigma(g_0, g_1) = \inf\{\epsilon > 0 \text{ such that } g_0(A) \leq g_1(A^\epsilon) + \epsilon \text{ for all closed } A \subset \mathbb{R}^3\};$$

we set $\Pi(g_0, g_1) = \max\{\sigma(g_0, g_1), \sigma(g_1, g_0)\}$. Here, we have denoted, abusing on the notation, the measure and its corresponding density, if any, by the same symbol.

b) the $(C^m)^*$ distance $\|g_0 - g_1\|_m^*$: for $m \geq 1$, let $C^m(\mathbb{R}^3)$ be the set of m-times continuously differentiable functions, endowed with its natural norm $\|\cdot\|_m$. Then, we consider its dual norm:

$$\|g_0 - g_1\|_m^* = \sup\left\{\left|\int_{\mathbb{R}^3} \varphi(v)(g_0(v) - g_1(v))\,dv\right|; \varphi \in C^m, \|\varphi\|_m \leq 1\right\}.$$

The following Lemma just summarizes the results proved in refs. 17, 21.

Lemma 4.9. ***Relation between probability metrics.*** *Given $g_0, g_1 \in \mathcal{S}_{\alpha,M}$, then there exist explicit constants C_1 and C_2 such that*

$$W_2^2(g_0, g_1) \leq (2M + 8)\Pi(g_0, g_1)^{\frac{\alpha}{\alpha+2}} + 4\Pi(g_0, g_1)^2,$$

$$\Pi(g_0, g_1) \leq \max\left\{C_1[\|g_0 - g_1\|_6^*]^{\frac{1}{7}}, \|g_0 - g_1\|_6^*\right\}$$

and

$$\|g_0 - g_1\|_6^* \leq C_2\left(\max\left\{\int_{\mathbb{R}^3} |v|^2 g_0\,dv, \int_{\mathbb{R}^3} |v|^2 g_1\,dv\right\}\right)^{\frac{4}{7}} d_2(g_0, g_1)^{\frac{1}{3}}.$$

With this lemma at hand and the uniform in time propagation of the fourth moment proved in Theorem 4.3, we deduce.

Corollary 4.10. ***(Exponential Decay of the Euclidean Wasserstein distance).*** *Given an initial datum $\hat{g}_0$ with unit mass, zero mean velocity, unit pressure tensor and fourth-order moment bounded. Then, there exist computable positive constants C_w and λ_w such that*

$$W_2(g(\tau), g_\infty) \leq C_w e^{-\lambda_w \tau},$$

for all $\tau \geq 0$.

5. BEYOND ERNST-BRITO CONJECTURE

We will study here some further properties of the self-similar solution to Eq. (2.4). In particular, we will derive its precise decay in velocity as $|v| \to \infty$, at least when the root $\alpha = \alpha_e$ where $G(\alpha, e) = 1$ is less than 2.

Let us choose the initial value like in Theorem 3.1, namely with unit pressure tensor. Its Fourier transform can be written as

$$\hat{f}_0(k) = 1 - \frac{1}{2}|k|^2 + \Psi_0(k)$$

with the rest $\Psi_0(k) = o(|k|^2)$. Likewise, since the momentum of the solution to Eq. (2.4) is identically equal to zero, and the temperature $\theta(\tau)$ evolves according to (2.5), we can write, at any subsequent time $\tau > 0$

$$\hat{f}(k, \tau) = 1 - \frac{1}{2}\theta(\tau)|k|^2 + \Psi(k, \tau)$$

with the rest $\Psi(k, \tau) = o(|k|^2)$. Properties in $|k| = 0$ of the remainder Ψ can be obtained by looking at the evolution of

$$D_\alpha(|k|, \tau) = \frac{\hat{f}(k, \tau) - 1 + \frac{1}{2}\theta(\tau)|k|^2}{|k|^{2+\alpha}}.$$

Theorem 5.1. ***(Non-strict contraction of the rest near zero).*** *Let $\hat{g}(k, \tau)$ be the solution of the time-scaled inelastic Maxwell equation (2.11) corresponding to the initial datum $\hat{f}(0)$ with unit mass, zero mean velocity and unit pressure tensor. Suppose in addition that $D_\alpha(|k|, \tau = 0) \le C$ for some value $0 < \alpha < 2$. Then, if $G(\alpha, e) \le 1$, for all $\tau > 0$ it holds*

$$\limsup_{\delta \to 0} \sup_{|k| \le \delta} \frac{|\hat{g}(k, \tau) - 1 + \frac{1}{2}|k|^2|}{|k|^{2+\alpha}} \le \limsup_{\delta \to 0} \sup_{|k| \le \delta} \frac{|\hat{f}_0(k) - 1 + \frac{1}{2}|k|^2|}{|k|^{2+\alpha}}.$$

Proof. Direct computations which use expressions (2.10) give

$$\frac{E}{4\pi}\int_{S^2} (|k^+|^2 + |k^-|^2 - |k|^2)\, dn = -2|k|^2.$$

Set $|k| < 1$. Considering that

$$\frac{\partial\left(1 - \frac{1}{2}\theta(\tau)|k|^2\right)}{\partial \tau} = -\frac{1}{2}|k|^2 \frac{d\theta}{d\tau} = \theta(\tau)|k|^2,$$

we obtain

$$\frac{\partial\left(1 - \frac{1}{2}\theta(\tau)|k|^2\right)}{\partial \tau} = \frac{E}{4\pi}\int_{S^2} \left\{ \left(1 - \frac{\theta(\tau)}{2}|k^+|^2\right)\left(1 - \frac{\theta(\tau)}{2}|k^-|^2\right) \right.$$

$$- \left(1 - \frac{\theta(\tau)}{2}|k|^2\right)\Bigg\} dn - \frac{E}{16\pi}\theta^2(\tau)\int_{S^2}|k^+|^2|k^-|^2 dn.$$

Proceeding now as in the proof of Theorem 3.1, we have

$$\left|\frac{\partial D_\alpha(|k|,\tau)}{\partial\tau} + E D_\alpha(|k|,\tau)\right| \le \frac{E}{4}\theta^2(\tau)|k|^{2-\alpha}$$
$$+\frac{E}{4\pi}\int_{S^2}\left[\frac{|k^-|^{2+\alpha}}{|k|^{2+\alpha}}|D_\alpha(|k^-|,\tau)| + \frac{|k^+|^{2+\alpha}}{|k|^{2+\alpha}}|D_\alpha(|k^+|,\tau)|\right] dn. \tag{5.1}$$

If $|k| \le \delta < 1$, and

$$D_\alpha(\tau) = \sup_{|k|\le\delta}|D_\alpha(|k|,\tau)|,$$

(5.1) implies

$$\left|\frac{\partial D_\alpha(|k|,\tau)}{\partial\tau} + E D_\alpha(|k|,\tau)\right| \le \frac{E}{4}\theta^2(\tau)\delta^{2-\alpha} + E A(\alpha,e) D_\alpha(\tau).$$

This is equivalent to

$$\left|\frac{\partial(D_\alpha(|k|,\tau)\mathrm{e}^{E\tau})}{\partial\tau}\right| \le \frac{E}{4}\theta^2(\tau)\mathrm{e}^{E\tau}\delta^{2-\alpha} + E A(\alpha,e) D_\alpha(\tau)\mathrm{e}^{E\tau}.$$

Integrating from 0 to τ we get

$$|D_\alpha(|k|,\tau)|\mathrm{e}^{E\tau} \le |D_\alpha(|k|,0)| + \int_0^\tau \left(\frac{E}{4}\theta^2(t)\mathrm{e}^{Et}\delta^{2-\alpha} + E A(\alpha,e) D_\alpha(t)\mathrm{e}^{Et}\right) dt.$$

Hence, if $H(\tau) = D_\alpha(\tau)\mathrm{e}^{E\tau}$,

$$H(\tau) \le H(0) + \int_0^\tau \frac{E}{4}\theta^2(t)\mathrm{e}^{Et}\delta^{2-\alpha} dt + \int_0^\tau E A(\alpha,e) H(t)\, dt.$$

Now, by applying the generalized Gronwall lemma we obtain

$$H(\tau) \le H(0)\mathrm{e}^{E A(\alpha,e)\tau} + \delta^{2-\alpha}\eta(\tau),$$

where $\eta(\tau)$ is bounded for all finite τ. Thus we have

$$D_\alpha(\tau) \le D_\alpha(0)\exp\{-E(1 - A(\alpha,e))\tau\} + \delta^{2-\alpha}\eta(\tau)\exp\{-E\tau\}.$$

If $\alpha < 2$, letting δ going to 0, we obtain

$$\lim_{\delta\to 0}\sup_{|k|\le\delta}\frac{|\hat f(k,\tau) - 1 + \frac{1}{2}\theta(\tau)|k|^2|}{|k|^{2+\alpha}} \le \lim_{\delta\to 0}\sup_{|k|\le\delta}\frac{|\hat f_0(k) - 1 + \frac{1}{2}|k|^2|}{|k|^{2+\alpha}}\mathrm{e}^{-E(1-A(\alpha,e))\tau}.$$

Since $\hat{g}(k,\tau) = \hat{f}\left(\frac{k}{\theta^{\frac{1}{2}}(\tau)}, \tau\right)$, we obtain for $\hat{g}$ the bound

$$\lim_{\delta\to 0}\sup_{|k|\le\delta} \frac{|\hat{g}(k,\tau) - 1 + \frac{1}{2}|k|^2|}{|k|^{2+\alpha}} \le \lim_{\delta\to 0}\sup_{|k|\le\delta} \frac{|\hat{f}_0(k) - 1 + \frac{1}{2}|k|^2|}{|k|^{2+\alpha}} \mathrm{e}^{-C(\alpha,e)\tau},$$

where $C(\alpha, e)$ is defined in (3.7). Thus, if $\alpha < 2$, and $G(\alpha, e) \le 1$, the theorem is proved. □

As a consequence of Theorem 5.1, letting $\tau \to \infty$, we obtain for the stationary state

$$\lim_{\delta\to 0}\sup_{|k|\le\delta} \frac{|\hat{g}_\infty(|k|) - 1 + \frac{1}{2}|k|^2|}{|k|^{2+\alpha}} \le C.$$

Thus, if $\alpha_e < 2$, we can write

$$\hat{g}_\infty(|k|) = 1 - \frac{1}{2}|k|^2 + \Psi(k), \tag{5.2}$$

and the remainder $\Psi(k)$ is such that

$$\lim_{|k|\to 0} \frac{\Psi(k)}{|k|^{2+p}} = 0, \quad p < \alpha_e.$$

As showed by Bobylev and Cercignani,[9] in scaled variables the self-similar solution satisfies the bounds

$$\exp\{-|k|^2\} \le |\hat{g}_\infty(|k|)| \le \exp\{-|k|\}(1+|k|). \tag{5.3}$$

In particular, the upper bound in (5.3) guarantees that the steady state $g_\infty(v)$ is smooth. In fact, by introducing the Sobolev space norms $\|\cdot\|_{H^r(\mathbb{R}^3)}$, $r \ge 0$ by

$$\|f\|^2_{H^r(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |k|^{2r} |\hat{f}(k)|^2 dk,$$

one sees at once that, for all $r > 0$,

$$\|g_\infty\|^2_{H^r(\mathbb{R}^3)} \le \int_{\mathbb{R}^3} |k|^{2r}(1+|k|)^2 \exp\{-2|k|\}\, dk < \infty.$$

Expansion (5.2), coupled with the regularity of the steady state, allows to apply the main theorem in Wong,[22] to conclude that the steady state $g_\infty(|v|)$ is given, for sufficiently large $|v|$, by the formula

$$g_\infty(|v|) = L(2)|v|^{-(5+\alpha_e)} + \delta(|v|),$$

where $L(2) = 2^{5/2}\Gamma[5/2]/\,\Gamma[-1]$, and the remainder δ is such that

$$\delta(|v|) = o(|v|^6) \quad \text{as } |v| \to \infty.$$

Remark 5.2. (Overpopulated tail behavior in velocity space). *This also solves a question posed by Ernst and Brito in refs. 15, 16 which, starting in $\mathbb{R}^d$ from an expansion of the type*

$$\hat{g}_\infty(|k|) = 1 - \frac{1}{2}|k|^2 + \sum_{l=2}^{[p]} C_l |k|^{2l} + B|k|^{2p}$$

arrived to the formal conclusion that

$$g_\infty(|v|) \cong |v|^{-(d+2p)}. \tag{5.4}$$

In [Ref. 9, Sec. 7], Bobylev and Cercignani outlined that conclusion (5.4), while quite probable, was difficult to prove. For enough regular functions g_∞, however, this conclusion is contained in ref. 22.

APPENDIX: PROOF OF THEOREM 4.4

Proof. Obviously it holds

$$d_{2+\alpha}(\hat{f}(\tau) - \hat{\Phi}(\tau), \hat{f}_{hc}(\tau)) \le \|h(k,\tau)\chi_{|k|\le 1}\|_\infty + \|h(k,\tau)\chi_{|k|>1}\|_\infty \tag{A.1}$$

where $\chi_{\mathcal{K}}$ denotes the characteristic function of the set $\mathcal{K}$ and $h(k,\tau)$ denotes

$$h(k,\tau) = \frac{\hat{f}(k,\tau) - \hat{\Phi}(k,\tau) - \hat{f}_{hc}(k,\tau)}{|k|^{2+\alpha}}.$$

At first, we focus our attention on the region $|k| \le 1$. Since both $\hat{f}(k,\tau)$ and $\hat{f}_{hc}(k,\tau)$ are solutions to the Boltzmann equation (2.9) we have

$$\begin{aligned} &\frac{\partial}{\partial\tau}\left(\hat{f}(k,\tau) - \hat{\Phi}(k,\tau) - \hat{f}_{hc}(k,\tau)\right) + E\left(\hat{f}(k,\tau) - \hat{\Phi}(k,\tau) - \hat{f}_{hc}(k,\tau)\right) \\ &= \frac{E}{4\pi}\int_{S^2}\left\{\hat{f}(k^+)\left[\hat{f}(k^-) - \hat{f}_{hc}(k^-)\right] + \hat{f}_{hc}(k^-)\left[\hat{f}(k^+) - \hat{f}_{hc}(k^+)\right]\right\} dn \\ &\quad - \frac{\partial\hat{\Phi}(k,\tau)}{\partial\tau} - E\,\hat{\Phi}(k,\tau)\,. \end{aligned} \tag{A.2}$$

Bearing in mind the definition of $\hat{\Phi}(k,\tau)$ given in (4.6) and the evolution law for the pressure tensor (4.5), we obtain

$$-\frac{\partial\hat{\Phi}(k,\tau)}{\partial\tau} - E\,\hat{\Phi}(k,\tau) = \frac{1}{2}E\sum_{i\neq j} p_{ij}(\tau)k_i k_j\left[1 - \frac{(1+e)(3-e)}{8}\right]. \tag{A.3}$$

So, if we set $\hat{F}(k,\tau) = \hat{f}(k,\tau) - \hat{\Phi}(k,\tau) - \hat{f}_{hc}(k,\tau)$, by substituting (A.3) into the equality (A.2) we find

$$\frac{\partial \hat{F}(k,\tau)}{\partial \tau} + E\hat{F}(k,\tau) = \frac{E}{4\pi}\int_{S^2}\left[\hat{f}(k^+)\hat{F}(k^-) + \hat{f}_{hc}(k^-)\hat{F}(k^+)\right]dn$$
$$+\frac{E}{2}\sum_{i\neq j} p_{ij}(\tau)k_ik_j\left[1 - \frac{(1+e)(3-e)}{8}\right]$$
$$+\frac{E}{4\pi}\int_{S^2}\left[\hat{f}(k^+)\hat{\Phi}(k^-) + \hat{f}_{hc}(k^-)\hat{\Phi}(k^+)\right]dn \quad \text{(A.4)}$$

Since

$$\hat{f}(k,\tau) = 1 - \frac{1}{2}\sum_{i,j=1}^{3} p_{ij}(\tau)k_ik_j + o(k^2)$$

while the similarity solution $f_{hc}(v,\tau)$ is such that

$$\hat{f}_{hc}(k,\tau) = 1 - \frac{1}{2}\sum_{i=1}^{3} p_{ii}(\tau)k_i^2 + o(k^2)\ ,$$

then the function in the last integral of (A.4) takes the form

$$\int_{S^2}\left[\hat{f}(k^+)\hat{\Phi}(k^-) + \hat{f}_{hc}(k^-)\hat{\Phi}(k^+)\right]dn = \int_{S^2}\left[\hat{\Phi}(k^-) + \hat{\Phi}(k^+)\right]dn$$
$$-\int_{S^2}\left\{\hat{\Phi}(k^-)\frac{1}{2}\sum_{i,j=1}^{3} p_{ij}(\tau)k_i^+k_j^+ + \hat{\Phi}(k^+)\frac{1}{2}\sum_{i=1}^{3} p_{ii}(\tau)(k_i^-)^2 + o(k^2)\right\}dn. \quad \text{(A.5)}$$

By resorting to the definitions (4.6) and (2.10), simple computations give

$$\frac{E}{4\pi}\int_{S^2}\left[\hat{\Phi}(k^-) + \hat{\Phi}(k^+)\right]dn = -\frac{E}{2}\sum_{i\neq j} p_{ij}(\tau)\left[\left(\frac{1+e}{4}\right)^2 + \left(\frac{3-e}{4}\right)^2\right]k_ik_j.$$

Concerning the last integral in (A.5), there exists $\bar{k}$ such that, if we denote

$$\bar{p}_{ij}(\tau) = \frac{\partial^2}{\partial k_i \partial k_j}\int_{\mathbb{R}^3} f(v,\tau)\mathrm{e}^{-ik\cdot v}dv\bigg|_{k=\bar{k}}$$

we have

$$\int_{S^2}\left\{\hat{\Phi}(k^-)\frac{1}{2}\sum_{i,j=1}^{3} p_{ij}(\tau)k_i^+k_j^+ + \hat{\Phi}(k^+)\frac{1}{2}\sum_{i=1}^{3} p_{ii}(\tau)(k_i^-)^2 + o(k^2)\right\}dn$$

$$= \int_{S^2} \left[\hat{\Phi}(k^-)\frac{1}{2}\sum_{i,j=1}^{3} \bar{p}_{ij}(\tau)k_i^+k_j^+ + \hat{\Phi}(k^+)\frac{1}{2}\sum_{i=1}^{3} \bar{p}_{ii}(\tau)(k_i^-)^2 \right] dn \ .$$

At the end, coming back to Eq. (A.4) we obtain

$$\frac{\partial \hat{F}(k,\tau)}{\partial t} + E\hat{F}(k,\tau) = \frac{E}{4\pi}\int_{S^2} \left[\hat{f}(k^+)\hat{F}(k^-) + \hat{f}_{hc}(k^-)\hat{F}(k^+) \right] dn$$

$$-\frac{E}{4\pi}\int_{S^2} \left[\hat{\Phi}(k^-)\frac{1}{2}\sum_{i,j=1}^{3} \bar{p}_{ij}(\tau)k_i^+k_j^+ + \hat{\Phi}(k^+)\frac{1}{2}\sum_{i=1}^{3} \bar{p}_{ii}(\tau)(k_i^-)^2 \right] dn. \quad \text{(A.6)}$$

From the definition of $\hat{\Phi}$ it follows that the last integral contains only terms of order $|k|^4$. Moreover, for each i, j the term $|\bar{p}_{ij}(\tau)|$ is bounded uniformly in time by the initial temperature. Therefore

$$\left| \hat{\Phi}(k^-)\frac{1}{2}\sum_{i,j=1}^{3} \bar{p}_{ij}(\tau)k_i^+k_j^+ \right| \leq \frac{1}{2}\sum_{i\neq j} |p_{ij}(\tau)||k_i^-k_j^-|\frac{1}{2}\theta_0 \sum_{i,j=1}^{3} |k_i^+k_j^+|$$

$$\leq \frac{\theta_0}{4}\left(\sum_{i\neq j}|k_i^-k_j^-|\right)\left(\sum_{i,j=1}^{3}|k_i^+k_j^+|\right)(\max_{i\neq j}|p_{ij}(0)|)\exp\left(-\frac{(1+e)(3-e)}{8}E\tau\right).$$

Straightforward estimates show

$$\left| \hat{\Phi}(k^-)\frac{1}{2}\sum_{i,j=1}^{3} \bar{p}_{ij}(\tau)k_i^+k_j^+ \right| \leq \frac{3}{2}|k|^4\theta_0(\max_{i\neq j}|p_{ij}(0)|)\exp\left(-\frac{(1+e)(3-e)}{8}E\tau\right)$$

and

$$\left| \hat{\Phi}(k^+)\frac{1}{2}\sum_{i=1}^{3} \bar{p}_{ii}(\tau)(k_i^-)^2 \right| \leq \frac{1}{2}|k|^4\theta_0(\max_{i\neq j}|p_{ij}(0)|)\exp\left(-\frac{(1+e)(3-e)}{8}E\tau\right)$$

that together with (A.6) imply that, for $|k| \leq 1$,

$$\left| \frac{\partial}{\partial\tau}\frac{\hat{F}(k,\tau)}{|k|^{2+\alpha}} + E\frac{\hat{F}(k,\tau)}{|k|^{2+\alpha}} \right| \leq E\ A(\alpha,e)\left(\sup_{|k|\leq 1}\frac{\hat{F}(k,\tau)}{|k|^{2+\alpha}}\right)$$

$$+2E\theta_0(\max_{i\neq j}|p_{ij}(0)|)|k|^{2-\alpha}\exp\left\{-\frac{(1+e)(3-e)}{8}E\tau\right\}.$$

Since $h(k,\tau) = \hat{F}(k,\tau)/|k|^{2+\alpha}$, Eq. (A.7) implies

$$\left|\frac{\partial}{\partial\tau}(h(k,\tau)\mathrm{e}^{E\tau})\right| \leq E\ A(\alpha,e)\|h(k,\tau)\chi_{|k|\leq 1}\|_\infty \mathrm{e}^{E\tau}$$

$$+ C\ \exp\left\{\left[1-\frac{(1+e)(3-e)}{8}\right]E\ \tau\right\}. \quad \text{(A.7)}$$

Integrating from 0 to τ, taking the supremum over k such that $|k| \leq 1$ and setting $z(\tau) = \|h(k,\tau)\chi_{|k|\leq 1}\|_\infty e^{E\tau}$ we have

$$z(\tau) \leq H(\tau) + E\, A(\alpha, e) \int_0^\tau z(s) ds$$

where

$$H(\tau) = \|h(k,0)\chi_{|k|\leq 1}\|_\infty + C \int_0^\tau \exp\left\{\left[1 - \frac{(1+e)(3-e)}{8}\right] Es\right\} ds.$$

Using the generalized Gronwall lemma, we get

$$z(\tau) \leq e^{E A(\alpha,e)\tau} \left\{ z(0) + C \int_0^\tau e^{\left[(1 - \frac{(1+e)(3-e)}{8} - A(\alpha,e))Es\right]} ds \right\}. \tag{A.8}$$

For $\alpha = 0$ we have $\frac{(1+e)(3-e)}{8} + A(\text{o}, e) = 1 + \frac{(e+1)^2}{8} > 1$, so we can choose $\bar{\alpha}$ close to zero in such a way that for $0 < \alpha \leq \bar{\alpha}$

$$\frac{(1+e)(3-e)}{8} + A(\alpha, e) \geq 1 + \frac{(e+1)^2}{16}.$$

In this interval

$$\int_0^\tau \exp\left[\left(1 - \frac{(1+e)(3-e)}{8} - A(\alpha,e)\right) Es\right] ds$$
$$\leq \int_0^\tau \exp\left(-\frac{1+e}{2(1-e)} s\right) ds \leq \frac{2(1-e)}{(1+e)} \qquad .$$

Substituting into (A.8) and reminding the definition of z, we get

$$\|h(k,\tau)\chi_{|k|\leq 1}\|_\infty \leq \left[\|h(k,0)\chi_{|k|\leq 1}\|_\infty + \frac{2C(1-e)}{1+e} \right] e^{-E(1-A(\alpha,e))\tau}. \tag{A.9}$$

As far as the region in which $|k| > 1$ is concerned, by means of analogous but much easier (since $\hat{\Phi} \equiv 0$) calculations we obtain the estimate

$$\|h(k,\tau)\chi_{|k|\leq 1}\|_\infty \leq \|h(k,0)\chi_{|k|>1}\|_\infty > e^{-E(1-A(\alpha,e))\tau}. \tag{A.10}$$

By inserting (A.9) and (A.10) into (A.1), the sought inequality (4.7) follows. □

ACKNOWLEDGEMENTS

The authors are grateful to Ugo Gianazza for stimulating discussions and for having drawn to their attention the result of paper 22 and to Irene Gamba for sharing with them the relevant new results in ref. 10 in draft form. JAC acknowledges the support from DGI-MEC (Spain) project MTM2005-08024. MB and GT acknowledge support from the Italian MIUR project "Mathematical Problems of Kinetic Theories." Part of this work was done while the first author and the third author were visiting the Department of Mathematics of the Autonoma University of Barcelona in December 2003 and January 2004 respectively, in the framework of the bilateral Project Italy-Spain Azioni Integrate. Supports from the European IHP network "Hyperbolic and Kinetic Equations: Asymptotics, Numerics, Applications" HPRN-CT-2002-00282 are also gratefully acknowledged.

REFERENCES

1. A. Baldassarri, U. M. B. Marconi, and A. Puglisi, "Influence of correlations in velocity statistics of scalar granular gases," *Europhys. Lett.* **58**:14–20 (2002).
2. E. Ben-Naim, and P. Krapivski, "Multiscaling in inelastic collisions," *Phys. Rev. E* **61**:R5–R8 (2000).
3. E. Ben-Naim, and P. Krapivski, "Nontrivial velocity distributions in inelastic gases," *J. Phys. A* **35**:L147–L152, (2002).
4. M. Bisi, J. A. Carrillo, and G. Toscani, "Contractive Metrics for a Boltzmann equation for granular gases: Diffusive equilibria," *J. Statist. Phys*. **118**:301–331 (2005).
5. A. V. Bobylev, "Exact solutions of the Boltzmann equation," *Doklady Akad. Nauk* SSSR **231**:571–574 (1975).
6. A. V. Bobylev, "The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules," *Sov. Sci. Rev. C. Math. Phys.* **7**:111–233 (1988).
7. A. V. Bobylev, J. A. Carrillo, and I. Gamba, "On some properties of kinetic and hydrodynamic equations for inelastic interactions," *J. Statist. Phys.* **98**:743–773 (2000).
8. A. V. Bobylev, and C. Cercignani, "Moment equations for a Granular Material in a Thermal Bath," *J. Statist. Phys.* **106**:547–567 (2002).
9. A. V. Bobylev, C. Cercignani, "Self-similar asymptotics for the Boltzmann equation with inelastic and elastic interactions," *J. Statist. Phys.* **110**:333–375 (2003).
10. A. V. Bobylev, C. Cercignani, and I. Gamba, "Generalized Maxwell models and self-similar asymptotics," personal communication and work in progress.
11. A. V. Bobylev, C. Cercignani, and G. Toscani, "Proof of an asymptotic property of self-similar solutions of the Boltzmann equation for granular materials," *J. Statist. Phys.* **111**:403–417 (2003).
12. A. V. Bobylev, and I. Gamba, "Boltzmann equations for mixtures of Maxwell gases: exact solutions and power-like tails," preprint 2005.
13. E. A. Carlen, E. Gabetta, and G. Toscani, "Propagation of smoothness and the rate of exponential convergence to equilibrium for a spatially homogeneous Maxwellian gas," *Commun. Math. Phys.* **305**:521–546 (1999).
14. J. A. Carrillo, R. J. McCann, and C. Villani, "Contractions in the 2-Wasserstein length space and thermalization of granular media," *Archive Rat. Mech. Anal.* **179**:247–293 (2006).
15. M. H. Ernst, and R. Brito, "High energy tails for inelastic Maxwell models," *Europhys. Lett.* **58**:182–187 (2002).
16. M. H. Ernst, and R. Brito, "Scaling solutions of inelastic Boltzmann equation with over-populated high energy tails," *J. Statist. Phys.* **109**:407–432 (2002).
17. E. Gabetta, G. Toscani, and W. Wennberg, "Metrics for Probability Distributions and the Trend to Equilibrium for Solutions of the Boltzmann Equation," *J. Statist. Phys.* **81**:901–934 (1995).
18. T. Krook, and T. T. Wu, "Formation of Maxwellian tails," *Phys. Rew. Lett.* **36**:1107–1109 (1976).
19. A. Pulvirenti, and G. Toscani, "Asymptotic properties of the inelastic Kac model," *J. Statist. Phys.* **114**:1453–1480 (2004).
20. H. Tanaka, "Probabilistic treatment of the Boltzmann equation of Maxwellian molecules," *Z. Wahrsch. Verw. Gebiete* **46**(1):67–105 (1978/79).

21. G. Toscani, and C. Villani, "Probability Metrics and Uniqueness of the Solution to the Boltzmann Equation for a Maxwell Gas," *J. Statist. Phys.* **94**:619–637 (1999).
22. R. Wong, "Distributional derivative of an asymptotic expansion," *Proceedings AMS* **80**:266–270 (1976).